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On Gromov's compactness Theorem

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Constant mean curvature surfaces in homogeneous manifolds

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These Lecture Notes report on a remarkable achievement of the last decade: the extension of this classical global theory to the case of CMC surfaces immersed in a homogeneous Riemannian 3-manifold. These homogeneous 3-spaces are the most *simple* and *symmetric* Riemannian 3-manifolds that one can consider apart from space forms, and are tightly related to the 3-dimensional Thurston geometries. In the simply connected case there is a complete list of these homogeneous 3-manifolds:

i) The canonical space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, whose isometry group is 6-dimensional.

ii) The product spaces $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg space Nil_3, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and the Berger spheres $\mathbb{S}^3_{\mathrm{berg}}$. We label these spaces as $\mathbb{E}^3(\kappa, \tau)$ spaces, as each of them can be expressed as a Riemannian submersion over a base surface $\mathbb{M}^2(\kappa)$ of constant curvature κ and bundle curvature τ. The isometry group of each $\mathbb{E}^3(\kappa, \tau)$ is 4-dimensional.

iii) Some Lie groups with a 3-dimensional isometry group. We may specially quote the space Sol_3, which is the only Thurston geometry among them.

The 3-dimensional Thurston geometries are modeled by 7 of these 8 types of manifolds: only Berger spheres are missing there, since their isometry group is contained in the isometry group of $\mathbb{S}^3$.

Despite some previous interesting works on the topic the study of CMC surfaces in homogeneous 3-manifolds started to develop as a consistent unified theory after a series of pioneering works by W.H. Meeks and H. Rosenberg and U. Abresch and H. Rosenberg. On the one hand, Meeks and Rosenberg explored the theory of complete minimal surfaces in product spaces $M^2 \times \mathbb{R}$, providing a basis for future research. On the other hand, Abresch and Rosenberg constructed a holomorphic quadratic differential for CMC surfaces in the spaces $\mathbb{E}^3(\kappa, \tau)$, which has attracted the attention of many researchers to this field. After just a few years of development, the subject has grown quickly in many different directions, and it already contains a large number of important contributions.

The aim of these notes is to present some problems at the core of the theory, and to explain the key ideas that have led to their complete or partial solution. Specifically, we shall focus on the following issues:

I. The presentation of the basic integrability equations of surface theory in homogeneous 3-manifolds from a unified point of view, and the geometric consequences that are deduced from there.

II. The problem of classifying compact CMC surfaces in homogeneous 3-manifolds that are either embedded or have genus zero. A central result for this will be the existence of the Abresch-Rosenberg differential referred to above. This problem is especially important, since isoperimetric regions in any Riemannian 3-manifold are bounded by compact embedded CMC surfaces.

III. The solution of the Bernstein problem for entire graphs of critical constant mean curvature (that is, the largest value of the mean curvature for which compact CMC surfaces fail to exist), and the derivation of several half-space theorems, also for surfaces of critical constant mean curvature.

IV. The study of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, including the construction of parabolic entire graphs and of examples with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, and of properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$.

On Gromov's Compactness Theorem

by Uwe Abresch

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This series of lectures is meant to be a introduction to a Gromov's compactness theorem for students that have taken the standard courses in Differential Geometry. In particular, we explain

i) the context in which the theorem first appeared,
ii) the precise statement of the theorem and its subtleties,
iii) the key ideas that are used in the proof of the theorem, and
iv) some basic applications.

After recalling the major results and tools in comparison geometry that were known at the time, we state Gromov's compactness theorem and describe its first application, namely how Berger used the theorem in order to extend his rigidity theorem into the pinching below $\frac{1}{4}$ theorem.

Then we state Cheeger's finiteness theorem, a result that is closely related to Gromov's compactness theorem. In fact, introducing what is now called the Gromov-Hausdorff distance d_{GH}, Gromov managed to generalize the standard compactness theorem for the space of all closed subsets X of a given compact metric space (Z, d) to a compactness theorem for suitable classes of compact metric spaces X. The arguments used in the proof of Cheeger's finiteness theorem the make it possible to pass from the rather coarse Gromov-Hausdorff topology to the much finer $\mathcal{C}^{1,\alpha}$-topologies. It is this step where things like ε-nets, relative injectivity radius estimates, harmonic coordinates, and the Riemannian center of math construction enter the picture.

We conclude this series of lectures describing how Gromov's compactness theorem can be used in order to solve geometric extremal value problems. The example that we chose for this purpose is the minimal volume metric on $\mathbb{R}^2$ constructed by Bavard and Pansu.

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